



19^η ΒΑΛΚΑΝΙΚΗ ΜΑΘΗΜΑΤΙΚΗ ΟΛΥΜΠΙΑΔΑ ΝΕΩΝ
24-29 Ιουνίου, 2015, Βελιγράδι, Σερβία

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Πρόβλημα 1. Βρείτε όλους τους πρώτους αριθμούς a, b, c και όλους τους θετικούς ακεραίους k που ικανοποιούν την εξίσωση

$$a^2 + b^2 + 16c^2 = 9k^2 + 1$$

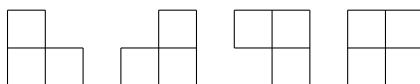
Πρόβλημα 2. Θεωρούμε τους θετικούς πραγματικούς αριθμούς a, b, c , που είναι τέτοιοι ώστε $a + b + c = 3$. Βρείτε την ελάχιστη τιμή της παράστασης

$$A = \frac{2 - a^3}{a} + \frac{2 - b^3}{b} + \frac{2 - c^3}{c}$$

Πρόβλημα 3. Δίνεται οξυγώνιο τρίγωνο ABC . Οι ευθείες l_1 και l_2 είναι κάθετες στην AB στα σημεία A και B , αντίστοιχα. Οι κάθετες ευθείες από το μέσον M του AB προς τις πλευρές AC και BC του τριγώνου τέμνουν τις ευθείες l_1 και l_2 στα σημεία E και F , αντίστοιχα. Αν D είναι το σημείο τομής των ευθειών EF και MC , να αποδείξετε ότι

$$\angle ADB = \angle EMF$$

Πρόβλημα 4. Καθένα από τα ακόλουθα τέσσερα σχήματα αποτελείται από τρία μοναδιαία τετράγωνα και καλείται L-σχήμα.



Δίνονται ένας 5×5 πίνακας, αποτελούμενος από 25 μοναδιαία τετράγωνα, ένας θετικός ακέραιος $k \leq 25$ και απεριόριστος αριθμός L-σχημάτων οποιουδήποτε τύπου. Δύο παίκτες, ο A και ο B , παίζουν το ακόλουθο παιχνίδι:

Ξεκινώντας με τον A , σημειώνουν εναλλάξ σε κάθε κίνησή τους ένα τετραγώνο που δεν είναι ήδη σημειωμένο, μέχρι να σημειώσουν συνολικά k μοναδιαία τετράγωνα.

Μία τοποθέτηση L-σχημάτων λέγεται *καλή* αν τα L-σχήματα δεν επικαλύπτονται και καθένα από αυτά καλύπτει ακριβώς τρία μοναδιαία τετράγωνα του πίνακα που δεν είναι σημειωμένα. Ο B κερδίζει αν μετά από οποιαδήποτε *καλή* τοποθέτηση L-σχημάτων μένουν ακάλυπτα τουλάχιστον 3 μοναδιαία τετράγωνα που δεν είναι σημειωμένα. Προσδιορίστε την ελάχιστη τιμή του k για την οποία ο B έχει στρατηγική νίκης.

Διάρκεια: 4 ώρες και 30 λεπτά
Κάθε πρόβλημα βαθμολογείται με 10 μονάδες



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Problem 1. Find all prime numbers a, b, c and positive integers k which satisfy the equation

$$a^2 + b^2 + 16 \cdot c^2 = 9 \cdot k^2 + 1.$$

Solution:

The relation $9 \cdot k^2 + 1 \equiv 1 \pmod{3}$ implies

$$a^2 + b^2 + 16 \cdot c^2 \equiv 1 \pmod{3} \Leftrightarrow a^2 + b^2 + c^2 \equiv 1 \pmod{3}.$$

Since $a^2 \equiv 0, 1 \pmod{3}$, $b^2 \equiv 0, 1 \pmod{3}$, $c^2 \equiv 0, 1 \pmod{3}$, we have:

a^2	0	0	0	0	1	1	1	1
b^2	0	0	1	1	0	0	1	1
c^2	0	1	0	1	0	1	0	1
$a^2 + b^2 + c^2$	0	1	1	2	1	2	2	0

From the previous table it follows that two of three prime numbers a, b, c are equal to 3.

Case 1. $a = b = 3$

We have

$$a^2 + b^2 + 16 \cdot c^2 = 9 \cdot k^2 + 1 \Leftrightarrow 9 \cdot k^2 - 16 \cdot c^2 = 17 \Leftrightarrow (3k - 4c) \cdot (3k + 4c) = 17,$$

$$\Leftrightarrow \begin{cases} 3k - 4c = 1, \\ 3k + 4c = 17, \end{cases} \Leftrightarrow \begin{cases} c = 2, \\ k = 3, \end{cases} \text{ and } (a, b, c, k) = (3, 3, 2, 3).$$

Case 2. $c = 3$

If $(3, b_0, c, k)$ is a solution of the given equation, then $(b_0, 3, c, k)$ is a solution too.

Let $a = 3$. We have

$$a^2 + b^2 + 16 \cdot c^2 = 9 \cdot k^2 + 1 \Leftrightarrow 9 \cdot k^2 - b^2 = 152 \Leftrightarrow (3k - b) \cdot (3k + b) = 152.$$

Both factors shall have the same parity and we obtain only 2 cases:

- $\begin{cases} 3k - b = 2, \\ 3k + b = 76, \end{cases} \Leftrightarrow \begin{cases} b = 37, \\ k = 13, \end{cases} \text{ and } (a, b, c, k) = (3, 37, 3, 13);$
- $\begin{cases} 3k - b = 4, \\ 3k + b = 38, \end{cases} \Leftrightarrow \begin{cases} b = 17, \\ k = 7, \end{cases} \text{ and } (a, b, c, k) = (3, 17, 3, 7).$

So, the given equation has 5 solutions:

$$\{(37, 3, 3, 13), (17, 3, 3, 7), (3, 37, 3, 13), (3, 17, 3, 7), (3, 3, 2, 3)\}.$$

Problem 2. Let a, b, c be positive real numbers such that $a + b + c = 3$. Find the minimum value of the expression

$$A = \frac{2 - a^3}{a} + \frac{2 - b^3}{b} + \frac{2 - c^3}{c}.$$



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Solution:

We can rewrite A as follows:

$$\begin{aligned} A &= \frac{2-a^3}{a} + \frac{2-b^3}{b} + \frac{2-c^3}{c} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - a^2 - b^2 - c^2 = \\ &2\left(\frac{ab+bc+ca}{abc}\right) - (a^2 + b^2 + c^2) = 2\left(\frac{ab+bc+ca}{abc}\right) - ((a+b+c)^2 - 2(ab+bc+ca)) = \\ &2\left(\frac{ab+bc+ca}{abc}\right) - (9 - 2(ab+bc+ca)) = 2\left(\frac{ab+bc+ca}{abc}\right) + 2(ab+bc+ca) - 9 = \\ &2(ab+bc+ca)\left(\frac{1}{abc} + 1\right) - 9. \end{aligned}$$

Recall now the well-known inequality $(x+y+z)^2 \geq 3(xy+yz+zx)$ and set $x=ab, y=bc, z=ca$, to obtain $(ab+bc+ca)^2 \geq 3abc(a+b+c) = 9abc$, where we have used $a+b+c=3$. By taking the square roots on both sides of the last one we obtain:

$$ab+bc+ca \geq 3\sqrt{abc}. \quad (1)$$

Also by using AM-GM inequality we get that

$$\frac{1}{abc} + 1 \geq 2\sqrt{\frac{1}{abc}}. \quad (2)$$

Multiplication of (1) and (2) gives:

$$(ab+bc+ca)\left(\frac{1}{abc} + 1\right) \geq 3\sqrt{abc} \cdot 2\sqrt{\frac{1}{abc}} = 6.$$

So $A \geq 2 \cdot 6 - 9 = 3$ and the equality holds if and only if $a=b=c=1$, so the minimum value is 3.

Problem 3. Let $\triangle ABC$ be an acute triangle. The lines l_1, l_2 are perpendicular to AB at the points A, B respectively. The perpendicular lines from the midpoint M of AB to the sides of the triangle AC, BC intersect the lines l_1, l_2 at the points E, F , respectively. If D is the intersection point of EF and MC , prove that

$$\angle ADB = \angle EMF.$$

Solution:

Let H, G be the points of intersection of ME, MF with AC, BC respectively. From the similarity of triangles $\triangle MHA$ and $\triangle MAE$ we get $\frac{MH}{MA} = \frac{MA}{ME}$, thus

$$MA^2 = MH \cdot ME. \quad (1)$$

Similarly, from the similarity of triangles $\triangle MBG$ and $\triangle MFB$ we get $\frac{MB}{MF} = \frac{MG}{MB}$, thus

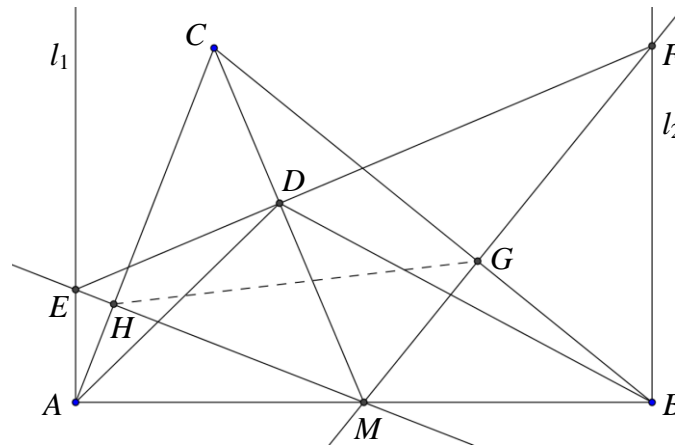
$$MB^2 = MF \cdot MG. \quad (2)$$

Since $MA = MB$, from (1), (2), we conclude that the points E, H, G, F are concyclic.



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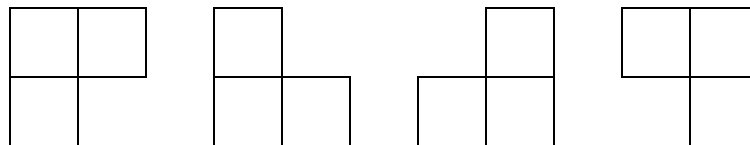
Therefore, we get that $\angle FEH = \angle FEM = \angle HGM$. Also, the quadrilateral $CHMG$ is cyclic, so $\angle CMH = \angle HGC$. We have

$$\angle FEH + \angle CMH = \angle HGM + \angle HGC = 90^\circ,$$

thus $CM \perp EF$. Now, from the cyclic quadrilaterals $FDMB$ and $EAMD$, we get that $\angle DFM = \angle DBM$ and $\angle DEM = \angle DAM$. Therefore, the triangles $\triangle EMF$ and $\triangle ADB$ are similar, so $\angle ADB = \angle EMF$.

Problem 4.

An L -figure is one of the following four pieces, each consisting of three unit squares:



A 5×5 board, consisting of 25 unit squares, a positive integer $k \leq 25$ and an unlimited supply L -figures are given. Two players, A and B , play the following game: starting with A they alternatively mark a previously unmarked unit square until they marked a total of k unit squares.

We say that a placement of L -figures on unmarked unit squares is called **good** if the L -figure do not overlap and each of them covers exactly three unmarked unit squares of the board. B wins if every **good** placement of L -figures leaves uncovered at least three unmarked unit squares. Determine the minimum value of k for which B has a winning strategy.

Solution:

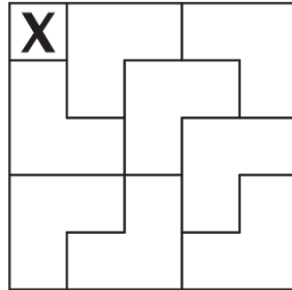
We will show that player A wins if $k = 1, 2, 3$, but player B wins if $k = 4$. Thus the smallest k for which B has a winning strategy exists and is equal to 4.

If $k = 1$, player A marks the upper left corner of the square and then fills it as follows.



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If $k = 2$, player **A** marks the upper left corner of the square. Whatever square player **B** marks, then player **A** can fill in the square in exactly the same pattern as above except that he doesn't put the L -figure which covers the marked square of **B**. Player **A** wins because he has left only two unmarked squares uncovered.

For $k = 3$, player **A** wins by following the same strategy. When he has to mark a square for the second time, he marks any yet unmarked square of the L -figure that covers the marked square of **B**.

Let us now show that for $k = 4$ player **B** has a winning strategy. Since there will be 21 unmarked squares, player **A** will need to cover all of them with seven L -figures. We can assume that in his first move, player **A** does not mark any square in the bottom two rows of the chessboard (otherwise just rotate the chessboard). In his first move player **B** marks the square labeled 1 in the following figure.

			1	4
		5	3	2

If player **A** in his next move does not mark any of the squares labeled 2, 3 and 4 then player **B** marks the square labeled 3. Player **B** wins as the square labeled 2 is left unmarked but cannot be covered with an L -figure.

If player **A** in his next move marks the square labeled 2, then player **B** marks the square labeled 5. Player **B** wins as the square labeled 3 is left unmarked but cannot be covered with an L -figure.

Finally, if player **A** in his next move marks one of the squares labeled 3 or 4, player **B** marks the other of these two squares. Player **B** wins as the square labeled 2 is left unmarked but cannot be covered with an L -figure.

Since we have covered all possible cases, player **B** wins when $k = 4$.