

**μ**  
**μ** **3**

**9 – 11 - 2015**

**μ**

)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $g: \mathbb{R} \rightarrow \mathbb{R}$ .  
 $(g \circ f)(x) = f(g(x))$

)  $f: (\alpha, \beta) \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{x}$ .

)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2$ .

)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) \neq 0$ ,  $x \in \Delta$ .

)  $\lim_{x \rightarrow x_0} |f(x)| = 0 \iff \lim_{x \rightarrow x_0} f(x) = 0$ .

)  $f(x) > 0$ ,  $x \in \Delta$ ,  $\lim_{x \rightarrow x_0} f(x) > 0$ .

)  $\lim_{x \rightarrow x_0} f(x) \leq \lim_{x \rightarrow x_0} g(x)$ ,  $x \in (\alpha, x_0) \cup (x_0, \beta)$ ,  $f(x) \leq g(x)$ .

)  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$ ,  $\lim_{x \rightarrow x_0} f(x) = 0$ .



)  $f: [\alpha, \beta] \rightarrow \mathbb{R}$ ,  $f(\alpha)f(\beta) > 0$ ,  $f(x) \neq 0$ ,  $x \in [\alpha, \beta]$ .

)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = 0$ .

)  $f: (\alpha, \beta) \rightarrow \mathbb{R}$ ,  $f(x) = 0$ ,  $x \in [\alpha, \beta]$ .

**μ)**  $f, g: \mathbb{R} \rightarrow \mathbb{R}$ .

)  $f(x) = \frac{1}{x}$ ,  $x \in \mathbb{R}^*$ .

)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x_1) = f(x_2)$ ,  $x_1, x_2 \in A_f$ .



μ 1,5x15

**μ**

$f(x) = \frac{\lambda x}{x-2}$ ,  $x \neq 2$ ,  $\lambda \in \mathbb{R}^*$ ,  $(f \circ f)(x) = x$

$x \neq 2$ .

)  $\lambda = 2$ ,  $f(x) = f^{-1}(x)$  μ 6

)  $f(x) = f^{-1}(x)$  μ 5

$x \neq 2$ .

)  $f(f(x+1)) + f(x+1) = x + \frac{2e^{-x}}{e^{-x}-2} + 1$ . μ 7

)  $f(x) = f^{-1}(x)$  μ 7

**μ**

$$f(x) = \frac{x^2 + \alpha x + \beta}{x - 2}, \alpha, \beta \in \mathbb{R} \quad \mu \quad \lim_{x \rightarrow 2} f(x) = 7.$$

)  $\alpha = 3 \quad \beta = -10.$  μ 4

)  $x_0 \in (0, 1), \quad f^2(x_0) = x_0^4 + x_0^3 + 6x_0 + 28.$  μ 5

)  $g: \mathbb{R} \rightarrow \mathbb{R} \quad f(g(x)) = \sqrt{x^2 + 1} - x + 5$   
 $x \in \mathbb{R}.$

i.  $g(x) = \sqrt{x^2 + 1} - x.$  μ 4

ii.  $g$  μ 5

iii.  $\lim_{x \rightarrow +\infty} (g(x+1) - g(x)) = 0.$  μ 4,5

iv.  $\sqrt{x^2 + 1} - \sqrt{10} < x - 3.$  μ 4



**μ**

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$f^2(x) - x^2 = e^{2x} + 2xf(x)$$

$x \in \mathbb{R} \quad f(0) = 1.$

)  $f(x) = e^x + x.$  μ 6

)  $\mu \quad x_0 \in \mathbb{R} \quad : e^{e^{x_0} + x_0} + e^{x_0} + x_0 = 1.$  μ 6

$$g: \mathbb{R} \rightarrow \mathbb{R}$$

$$e^{g^{\theta} x} > e^{y-x} \quad \forall y-x > g^{\theta} x:$$

$x \in \mathbb{R}.$

)  $g(x) = \eta \mu x.$  μ 5

) :

i.  $\lim_{x \rightarrow +\infty} \frac{x - g(x)}{x + 1}$

ii.  $\lim_{x \rightarrow +\infty} \left[ x^2 g\left(\frac{1}{x}\right) \right]$

iii.  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{2 - x}{1 - g(x)}$

μ 3+3+3



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**Ασκησόπολις**  
ο πιο πλούσιος κόσμος  
θεμάτων και ασκήσεων



$$) f(x) = \frac{2x}{x-2} = \frac{2x-4+4}{x-2} = 2 + \frac{4}{x-2}$$

$$x_1 < x_2 < 2 \text{ τότε } x_1 - 2 < x_2 - 2 < 0 \Leftrightarrow \frac{1}{x_1 - 2} > \frac{1}{x_2 - 2} \Leftrightarrow$$

$$\frac{4}{x_1 - 2} > \frac{4}{x_2 - 2} \Leftrightarrow 2 + \frac{4}{x_1 - 2} > 2 + \frac{4}{x_2 - 2} \Leftrightarrow f(x_1) > f(x_2) \Rightarrow f \searrow (-\infty, 2)$$

$$2 < x_1 < x_2 \text{ τότε } 0 < x_1 - 2 < x_2 - 2 \Leftrightarrow \frac{1}{x_1 - 2} > \frac{1}{x_2 - 2} \Leftrightarrow$$

$$\frac{4}{x_1 - 2} > \frac{4}{x_2 - 2} \Leftrightarrow 2 + \frac{4}{x_1 - 2} > 2 + \frac{4}{x_2 - 2} \Leftrightarrow f(x_1) > f(x_2) \Rightarrow f \searrow (2, +\infty)$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{2x}{x-2} = \lim_{x \rightarrow -\infty} \frac{2\cancel{x}}{\cancel{x}} = 2, \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{2x}{x-2} = \lim_{x \rightarrow +\infty} \frac{2\cancel{x}}{\cancel{x}} = 2.$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \left( 2x \frac{1}{x-2} \right) = -\infty \quad \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \left( 2x \frac{1}{x-2} \right) = +\infty$$

$$\mu \quad \Delta_1 = (-\infty, 2) \quad f$$

$$f(\Delta_1) = \left( \lim_{x \rightarrow 2^-} f(x), \lim_{x \rightarrow -\infty} f(x) \right) = (-\infty, 2).$$

$$\mu \quad \Delta_2 = (2, +\infty) \quad f$$

$$f(\Delta_2) = \left( \lim_{x \rightarrow +\infty} f(x), \lim_{x \rightarrow 2^+} f(x) \right) = (2, +\infty). \quad f(A) = (-\infty, 2) \cup (2, +\infty) = \mathbb{R} - \{2\}.$$



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$$) \quad x \neq 2 \quad f(x) = \frac{x^2 + \alpha x + \beta}{x-2} \Leftrightarrow f(x)(x-2) = x^2 + \alpha x + \beta$$

$$\lim_{x \rightarrow 2} [f(x)(x-2)] = \lim_{x \rightarrow 2} (x^2 + \alpha x + \beta) \Leftrightarrow 0 = 4 + 2\alpha + \beta \Leftrightarrow \beta = -4 - 2\alpha \quad (1)$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 + \alpha x + \beta}{x-2} = \lim_{x \rightarrow 2} \frac{x^2 + \alpha x - 4 - 2\alpha}{x-2} = \lim_{x \rightarrow 2} \frac{\cancel{(x-2)}(x+2) + \alpha \cancel{(x-2)}}{\cancel{x-2}} = 4 + \alpha$$

$$\lim_{x \rightarrow 2} f(x) = 7 \quad 4 + \alpha = 7 \Leftrightarrow \alpha = 3, \quad (1) \Rightarrow \beta = -10.$$

$$) \quad \alpha = 3 \quad \beta = -10 \quad f(x) = \frac{x^2 + 3x - 10}{x-2} = \frac{\cancel{(x-2)}(x+5)}{\cancel{x-2}} = x + 5$$

$$h(x) = x^4 + x^3 + 6x + 28 - f^2(x) = x^4 + x^3 + 6x + 28 - (x+5)^2 \Leftrightarrow$$

$$h(x) = x^4 + x^3 + 6x + 28 - x^2 - 10x - 25 = x^4 + x^3 - x^2 - 4x + 3 = (x-1)(x^3 + 2x^2 + x - 3)$$

$$\varphi(x) = x^3 + 2x^2 + x - 3, \quad x \in [0, 1].$$

$$\varphi(0) = -3, \varphi(1) = 1, \quad \varphi(0)\varphi(1) < 0$$

$$\mu \quad , \quad \mu \quad \text{Bolzano}, \quad x_0 \in (0, 1) \quad ,$$

$$\varphi(x_0) = 0 \Leftrightarrow x_0^3 + 2x_0^2 + x_0 - 3 = 0 \Rightarrow h(x_0) = 0 \Leftrightarrow f^2(x_0) = x_0^4 + x_0^3 + 6x_0 + 28.$$

$$) \text{ i. } f(g(x)) = \sqrt{x^2 + 1} - x + 5 \Leftrightarrow g(x) + 5 = \sqrt{x^2 + 1} - x + 5 \Leftrightarrow g(x) = \sqrt{x^2 + 1} - x$$

$$\text{ii. } f(x_1) - f(x_2) = \sqrt{x_1^2 + 1} - x_1 - \sqrt{x_2^2 + 1} + x_2 = \frac{x_1^2 + 1 - x_2^2 - 1}{\sqrt{x_1^2 + 1} + \sqrt{x_2^2 + 1}} - (x_1 - x_2) \Leftrightarrow$$

$$f(x_1) - f(x_2) = \frac{(x_1 - x_2)(x_1 + x_2)}{\sqrt{x_1^2 + 1} + \sqrt{x_2^2 + 1}} - (x_1 - x_2) = (x_1 - x_2) \left[ \frac{x_1 + x_2}{\sqrt{x_1^2 + 1} + \sqrt{x_2^2 + 1}} - 1 \right] =,$$

$$(x_1 - x_2) \left[ \frac{x_1 + x_2 - \sqrt{x_1^2 + 1} - \sqrt{x_2^2 + 1}}{\sqrt{x_1^2 + 1} + \sqrt{x_2^2 + 1}} \right]$$

Είναι  $\sqrt{x^2 + 1} > \sqrt{x^2} = |x| \Leftrightarrow -\sqrt{x^2 + 1} < x < \sqrt{x^2 + 1} \Rightarrow x - \sqrt{x^2 + 1} < 0$ , άρα

$x_1 - \sqrt{x_1^2 + 1} < 0$ ,  $x_2 - \sqrt{x_2^2 + 1} < 0$  και επειδή  $x_1 - x_2 < 0$  είναι

$f(x_1) - f(x_2) > 0 \Leftrightarrow f(x_1) > f(x_2) \Rightarrow f \nearrow \mathbb{R}$ .

$$\text{iii. } \lim_{x \rightarrow +\infty} (g(x+1) - g(x)) = \lim_{x \rightarrow +\infty} (\sqrt{(x+1)^2 + 1} - x - 1 - \sqrt{x^2 + 1} + x) =$$

$$\lim_{x \rightarrow +\infty} (\sqrt{x^2 + 2x + 2} - \sqrt{x^2 + 1} - 1) = \lim_{x \rightarrow +\infty} \left( \frac{x^2 + 2x + 2 - x^2 - 1}{\sqrt{x^2 + 2x + 2} + \sqrt{x^2 + 1}} - 1 \right) =$$

$$\lim_{x \rightarrow +\infty} \left( \frac{x \left( 2 + \frac{1}{x} \right)}{x \left( \sqrt{1 + \frac{2}{x} + \frac{2}{x^2}} + \sqrt{1 + \frac{1}{x^2}} \right)} - 1 \right) = \frac{2}{1+1} - 1 = 0$$



$$\text{iv. } \sqrt{x^2 + 1} - \sqrt{10} < x - 3 \Leftrightarrow \sqrt{x^2 + 1} - x < \sqrt{10} - 3 \Leftrightarrow g(x) < g(3) \stackrel{g \searrow}{\Leftrightarrow} x > 3.$$

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$$) f^2(x) - x^2 = e^{2x} + 2xf(x) \Leftrightarrow f^2(x) - 2xf(x) + x^2 = e^{2x} \Leftrightarrow (f(x) - x)^2 = e^{2x} \quad (1)$$

$$e^{2x} \neq 0 \quad h(x) = f(x) - x \neq 0$$

$$\mu \cdot h(0) = f(0) = 1 > 0, \quad h(x) > 0 \quad x \in \mathbb{R}, \quad (1) \quad :$$

$$f(x) - x = e^x \Leftrightarrow f(x) = e^x + x.$$

$$) \quad x_1, x_2 \in \mathbb{R} \quad \mu \quad x_1 < x_2, \quad e^{x_1} < e^{x_2} \quad e^{x_1} + x_1 < e^{x_2} + x_2 \Leftrightarrow$$

$$f(x_1) < f(x_2) \Rightarrow f \nearrow \mathbb{R} \Rightarrow f \nearrow \mathbb{R}.$$

$$e^{e^{x_0} + x_0} + e^{x_0} + x_0 = 1 \Leftrightarrow e^{f(x_0)} + f(x_0) = f(0) \Leftrightarrow f(f(x_0)) = f(0) \Leftrightarrow f(x_0) = 0.$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} (e^x + x) = -\infty \quad \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} (e^x + x) = +\infty.$$

$$f \quad \mu \quad f(A) = \left( \lim_{x \rightarrow -\infty} f(x), \lim_{x \rightarrow +\infty} f(x) \right) = \mathbb{R}.$$

$$0 \in f(A) \quad \mu \quad x_0 \in \mathbb{R}, \quad f(x_0) = 0.$$



$$) e^{g(x)} - e^{\eta \mu x} = \eta \mu x - g(x) \Leftrightarrow e^{g(x)} + g(x) = e^{\eta \mu x} + \eta \mu x \Leftrightarrow f(g(x)) = f(\eta \mu x) \stackrel{f \nearrow}{\Leftrightarrow} g(x) = \eta \mu x.$$

$$\text{ii. } \lim_{x \rightarrow +\infty} \frac{x - g(x)}{x + 1} = \lim_{x \rightarrow +\infty} \frac{x - \eta \mu x}{x + 1} = \lim_{x \rightarrow +\infty} \frac{x \left( 1 - \frac{\eta \mu x}{x} \right)}{x \left( 1 + \frac{1}{x} \right)} = 1 \quad x \in (0, +\infty)$$

$$\left| \frac{\eta\mu x}{x} \right| = \frac{|\eta\mu x|}{|x|} \leq \frac{1}{x} \Leftrightarrow -\frac{1}{x} \leq \frac{\eta\mu x}{x} \leq \frac{1}{x} \quad . \quad \lim_{x \rightarrow +\infty} \frac{\eta\mu x}{x} = 0.$$

$$\text{ii. } \lim_{x \rightarrow +\infty} \left[ x^2 g\left(\frac{1}{x}\right) \right] = \lim_{x \rightarrow +\infty} \left( x^2 \eta\mu \frac{1}{x} \right) \stackrel{\frac{1}{x}=u}{=} \lim_{u \rightarrow 0^+} \left( \frac{1}{u^2} \eta\mu u \right) = \lim_{u \rightarrow 0^+} \left( \frac{1}{u} \frac{\eta\mu u}{u} \right) = +\infty.$$

$$\text{iii. } \lim_{x \rightarrow \frac{\pi}{2}} \frac{2-x}{1-g(x)} = \lim_{x \rightarrow \frac{\pi}{2}} \left[ (2-x) \frac{1}{1-\eta\mu x} \right] = +\infty \quad \lim_{x \rightarrow \frac{\pi}{2}} (2-x) = 2 - \frac{\pi}{2} > 0$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{1-\eta\mu x} = +\infty \quad \lim_{x \rightarrow \frac{\pi}{2}} (1-\eta\mu x) = 0 \quad 1-\eta\mu x > 0 \quad x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

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